THE PROPAGATION OF SHOCK WAVES IN A SEMIBOUNDED VOLUME

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When a shock wave emerges from a pipe situated in a semibounded volume a system of waves arises between the end of the pipe and the bottom of the volume, and also in the gap between the pipe and the side walls of the volume. Paper [1] considers the propagation of a shock wave after emerging from the pipe until touching the side walls of the volume. The present paper considers the gas motion in a semibounded volume after the shock wave has traversed the volume and made contact with the side walls.

In part 1 a physical model is constructed of the gas motion up to the time when the primary shock wave reaches the bottom of the volume. In part 2 relations are found which enable us to determine the stream parameters in the semibounded volume up to the time when the primary shock wave arrives at the bottom of the volume. Section 3 considers the motion of the reflected shock wave between the pipe and the side walls of the volume.

NOTATION

$N = Na_1$ is the velocity of propagation of the shock wave; $u = ua_1$ is the velocity of gas particles; $\rho = \rho\rho_1$ is the gas density; $p = pp_1$ is the pressure; $M = u/a$ is the Mach number; $\lambda = u/a$ is the reduced velocity; $a = aa_1$ is the speed of sound propagation; $k = c_p/c_v$ is the adiabatic index; $t = a_1 t/d$ is the dimensionless time; $F = F/d^2$ is the area; $V = V/d^3$ is the volume; d is the inside diameter of pipe [m]; t = time [sec].

Subscript 1 denotes gas parameters in the volume before the shock wave has emerged from the pipe, subscript 2 denotes gas parameters behind the shock wave front in the pipe, an overscore denotes dimensionless parameters, while subscript 0 denotes parameters in the adiabatically decelerated gas; an asterisk denotes critical stream parameters.

1. A pipe is situated in a semibounded volume e (Fig. 1). A shock wave propagates along the pipe with a velocity N_1, and the gas parameters are constant behind the shock wave front.

The following cases of stream formation may occur depending on the stream parameters in the pipe behind the shock wave front, and on the areas F, F_1, F_2.

(a) The velocity $M_2 \geq 1$, of the wake behind the shock-wave front in the pipe and the value of the areas F, F_1, F_2 are such that the pressure in the end plane of the pipe is $p_4 > 1$, as a result of gas expansion behind the shock wave. We assume that the pressure on both sides of the end plane of the pipe is the same, and that the division of the stream emerging from the tube to the left, into the gap, and to the right, occurs in the end plane of the pipe. It should be noted that the assumption that the pressure is the same on both sides of the end plane of the pipe may be replaced by any other picture of the flow pattern in this zone. The assumption which has been adopted is justified by the good agreement between calculated values of the stream parameters and experiment performed in the Laboratory of Gasdynamics of Leningrad State University. Since $p_4 > 1$, a shock wave will propagate with a velocity N_{11} in the gap between the pipe and the side walls of the volume (Fig. 1a). Since the velocity of the stream emerging from the pipe is supersonic, there is a velocity increase when the stream expands in the volume, while the pressure decreases. On the other hand, when the area of the shock wave in creases from the value F to F_1 on emerging from the pipe, the pressure and gas velocity at the shock wave front decrease.

In order to avoid the resulting contradiction we must postulate the presence of a secondary shock wave between the end of the pipe and the primary shock wave passing out of the pipe. The secondary shock wave will propagate relative to the gas particles which have left the pipe in a direction opposite to their motion. The gas stream will suffer a braking effect on passing through the secondary shock wave. The gas in the region between the primary and secondary shock waves is divided by a contact surface (steady-state discontinuity). Gas which has emerged from the pipe is on the left of the contact surface, while gas which has been in the volume is on the right.

(b) The velocity $M_2 \geq 1$ of the wake behind the shock wave will propagate in the gap. The gas-flow config- F_1, F_2 is such that the pressure in the end plane of the pipe is $p_4 < 1$ as a result of gas expansion behind the shock wave. We assume, as before, that the pressure on both sides of the end plane of the pipe is the same and that the streams from the gap and from the pipe mix in the end plane of the pipe. A rarefaction wave will propagate in the gap. The gas flow configuration is represented in Fig. 1b.

It may be shown by arguments similar to those of (a) that a primary shock wave, a contact surface, and a secondary shock wave will propagate in the region between the end of the pipe and the bottom of the volume.

(c) The velocity $M_2 \leq 1$ of the wake behind the snock-wave front in the pipe and the value of the areas F, F_1, F_2 is such that the gas is accelerated to a value $M_3 = 1$ in the rarefaction wave which has passed into the pipe. Because of gas expansion behind the shock wave which has passed into the volume, the pressure in the end plane of the pipe may be either $p_4 > 1$, or $p_4 < 1$ as in cases (a) and (b) previously considered. The flow pattern in the semibounded volume will then be the same as in (a) or (b), respectively.

In (a), (b), and (c) values of the initial data N_1, F, F_1, F_2 were considered for the case in which the stream velocity in the volume in front of the secondary shock wave front is higher than the propagation velocity of the secondary shock wave. The secondary shock wave propagates down stream relative to the walls of the volume.

(d) As the area F_1 increases the stream velocity in front of the secondary shock wave increases more slowly than the velocity of the secondary shock wave.

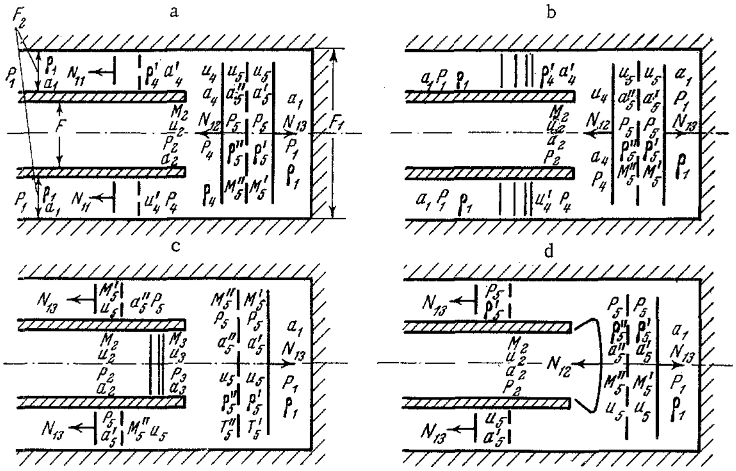

Fig. 1. Gas flow pattern in a semibounded volume for various values of the shock wave intensity in the pipe and for various values of the areas; a) for $M_2 \geq 1$, $p_4 > 1$; b) for $M_2 \geq 1$, $p_4 < 1$; c) for $M_3 < 1$, $p_5 > 1$; d) for $M_4 > N_{12}/a_4$.

There will come a time when the stream velocity will be equal to the secondary shock wave velocity, and then the secondary shock wave will not move relative to the walls of the volume. As F_1 increases further the boundaries of the secondary shock wave break away from the side walls of the volume; this wave is then situated in the immediate locality of the end of the pipe. A shock wave of the same intensity as the original shock wave then propagates in the gap. The flow configuration is represented in Fig. 1d.

(e) The wake velocity behind the shock wave front in the pipe is $M_2 < 1$, and the values of the areas F, F_1, F_2 such that the gas in the rarefaction wavewhich has passed into the pipe is accelerated to $M_3 < 1$. In this case $p_4 > 1$, there is no secondary shock wave, and a shock wave propagates in the gap of the same intensity as the primary shock wave in the region between the end of the pipe and the bottom of the volume (flow pattern as in Fig. 1c).

2. (a) The equations of conservation of mass, momentum, and energy for the control surfaces situated at the open end of the pipe, in the gap between the end of the pipe and the shock wave, and between the secondary shock wave and the end of the pipe have the form

$$u_2\rho_2 F = u_4\rho_4 F_1 + u_4'\rho_4' F_2,$$

$$\rho_2 u_2^2 F + p_2 F = \rho_4 u_4^2 F_1 + \rho_4' u_4'^2 F_2 + p_4 F_2 + p_4 F_1,$$

$$\rho_2 u_2 F\left(\frac{1}{k-1}\frac{p_2}{\rho_2} + \frac{u_2^2}{2}\right) = \rho_4 u_4 F_1\left(\frac{1}{k-1}\frac{p_4}{\rho_4} + \frac{u_4^2}{2}\right) +$$

$$+ \rho_4' u_4' F_2\left(\frac{1}{k-1}\frac{p_4}{\rho_4'} + \frac{u_4'^2}{2}\right). \tag{2.1}$$

It is assumed that the gas in front of the secondary shock wave propagates isentropically. Figure 1a should be consulted for the notation employed. Using the condition of dynamic compatibility at the shock-wave in the gap we can write out relations expressing u_4' and ρ_4' in terms of p_4:

$$u_4' = (p_4 - 1) f, \qquad \rho_4' = \frac{\varkappa p_4 + 1}{\varkappa + p_4},$$

$$f = \frac{1}{k}\left(\frac{k+1}{2k} p_4 + \frac{k-1}{2k}\right)^{-1/2}, \quad \varkappa = \frac{k+1}{k-1}. \tag{2.2}$$

Solving the system of equations (2.1), (2.2) for p_4, we obtain

$$qe = F_2 f(p_4 - 1)\left(\frac{p_4}{k-1} + b\right) +$$

$$+ \frac{[m - (F_1 - F_2) p_4 - 2F_2 b] F_1}{q - F_2 cf} \times$$

$$\times \left\{\frac{p_4}{k-1} + \frac{1}{2F_1}[m - (F_1 - F_2) p_4 - 2F_2 b]\right\},$$

$$q = \rho_2 u_2 F, \qquad e = \frac{1}{k-1}\frac{p_2}{\rho_2} + \frac{u_2^2}{2},$$

$$m = \rho_2 u_2^2 F + p_2 F, \qquad b = \frac{(p_4 - 1)^2}{k[(k+1) + (k-1) p_4]},$$

$$c = \frac{(p_4 - 1)[(k+1) p_4 + (k-1)]}{[(k+1) + (k-1) p_4]},$$

$$u_4 = \frac{m - (F_1 - F_2) p_4 - 2F_2 b}{q - F_2 cf},$$

$$\rho_4 = \frac{(q - F_2 cf)^2}{F_1[m - (F_1 - F_2) p_4 - 2F_2 b]}. \tag{2.3}$$

The relations cited allow us to determine the stream parameters in the region between the shock wave in the gap and the secondary shock wave.

Using the condition of dynamic compatibility at the primary and secondary shock waves we obtain

$$[(\gamma + 1) N_{13}^2 - \gamma] = p_4[(\gamma + 1) N_{12}^2 / a_4^2 - \gamma],$$

$$2(N_{13} - 1/N_{13})/(k+1) =$$

$$= u_4 - 2(N_{12} - a_4^2/N_{12})/(k+1),$$

$$\gamma = (k-1)/(k+1). \tag{2.4}$$

We now determine the values of N_{13} and N_{12} by solving the system of equations (2.4). The remaining stream parameters in the region between the primary and secondary shock waves can be found from dynamic compatibility conditions.

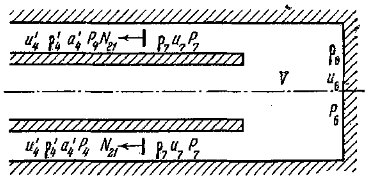

Fig. 2. Gas flow pattern in a semi-bounded volume after the reflected shock wave has passed into the gap.

(b) The equations of conservation of mass, momentum, and energy for control surfaces situated at the outlet from the open end of the pipe, between the secondary shock wave and the end of the pipe, have the form

$$u_2\rho_2 F = u_4\rho_4 F_1 - u_4'\rho_4' F_2,$$

$$\rho_2 u_2^2 F + p_2 F = \rho_4 u_4^2 F_1 - \rho_4' u_4'^2 F_2 + p_4 F_1 - p_4 F_2,$$

$$\rho_2 u_2 F\left(\frac{1}{k-1}\frac{p_2}{\rho_2} + \frac{u_2^2}{2}\right) =$$

$$= \rho_4 u_4 F_1\left(\frac{1}{k-1}\frac{p_4}{\rho_4} + \frac{u_4^2}{2}\right) - \rho_4' u_4' F_2\left(\frac{1}{k-1}\frac{p_4}{\rho_4'} + \frac{u_4'^2}{2}\right). \quad (2.5)$$

It is assumed that the gas expands isentropically up to the secondary shock wave. The notation employed is given in Fig. 1b.

At the rarefaction wave we have

$$u_4' = \frac{2}{k-1}\left(1 - p_4^{1/2(k-1)/k}\right), \qquad \rho_4' = p_4^{1/k}. \quad (2.6)$$

Solving Eqs. (2.5) and (2.6) for p_4 gives

$$qe = \frac{[m + F_2 h - (F_1 - F_2) p_4] F_1}{q + F_2 n}\left\{\frac{p_4}{k-1} - \frac{1}{2F_1}[m +\right.$$

$$\left. + F_2 h - (F_1 - F_2) p_4]\right\} - F_2 n p_4^{-1/k}\left(\frac{p_4}{k-1} + \frac{F_2 h}{2}\right),$$

$$h = \frac{4}{(k-1)^2}\left(1 - p_4^{1/2(k-1)/k}\right)^2 p_4^{1/k},$$

$$n = \frac{2}{k-1}\left(1 - p_4^{1/2(k-1)/k}\right) p_4^{1/k},$$

$$u_4 = \frac{m + F_2 h - (F_1 - F_2) p_4}{q + F_2 n},$$

$$\rho_4 = \frac{(q + F_2 n)^2}{F_1 [m + F_2 h - (F_1 - F_2) p_4]}. \quad (2.7)$$

These relations together with Eqs. (2.4) enable us to determine all the stream parameters in the region between the primary shock wave and the rarefaction wave.

In case (c) using the condition $M_3 = 1$, the flow parameters at the pipe outlet may be found from the familiar relations for a rarefaction wave. The form of the equations determining the flow parameters in the volume will be the same as in cases (a) and (b). Instead of initial flow parameters denoted in (a) and (b) by the subscript 2, we introduce initial parameters denoted by the subscript 3, which hold on leaving the rarefaction wave on condition that $M_3 = 1$.

(c) Expressing the values of the terms in the obvious identity

$$p_5 = \frac{p_5}{p_{05}}\frac{p_{05}^*}{p_{04}}\frac{p_{02}}{p_2} p_2,$$

as functions of λ and N_{13}, we obtain

$$[(\gamma + 1) N_{13}^2 - \gamma] = \left(\frac{1 - \gamma\lambda_5^2}{1 - \gamma\lambda_2^2}\right)^{k/(k-1)} \times$$

$$\times \left(\frac{1 - \gamma\lambda_4^2}{1 - \gamma/\lambda_4^2}\right)^{1/(k-1)} [(\gamma + 1) N_1^2 - \gamma]\lambda_4^2, \quad (2.8)$$

where the notation is in accordance with the flow configuration (see Fig. 1d.)

Using the condition of dynamic compatibility to express the value of u_5 in the identity $u_5 = \lambda_5'' a_{*5}''$, and a_{*5}'' in terms of a_2 and λ_2, we obtain

$$\frac{2}{k+1}\left(N_{13} - \frac{1}{N_{13}}\right) = \lambda_5''\left(\frac{2}{k-1}\right)^{1/2}\frac{a_2}{(1 - \lambda_2^2\gamma)^{1/2}}. \quad (2.9)$$

It is assumed that the stream behind the secondary shock wave in front of the contact surface expands isentropically to an area equal to the sum of the transfer cross sections $(F_1 + F_2)$. If F denotes the surface area of the secondary shock wave we have from the equation of continuity

$$\frac{F}{F_*} = \frac{\lambda_4}{\lambda_2}\left(\frac{1 - \lambda_4^2\gamma}{1 - \lambda_2^2\gamma}\right)^{1/(k-1)},$$

$$\frac{F_*}{F_1 + F_2} = \lambda_5''\lambda_4\left(\frac{1 - \lambda_5''^2\gamma}{1 - \gamma/\lambda_4^2}\right)^{1/(k-1)}.$$

Thus

$$\frac{F}{F_1 + F_2} = \frac{\lambda_4^2\lambda_5''}{\lambda_2}\left[\frac{(1 - \lambda_4^2\gamma)(1 - \lambda_5''^2\gamma)}{(1 - \lambda_2^2\gamma)(1 - \gamma/\lambda_4^2)}\right]^{1/(k-1)}. \quad (2.10)$$

Using (2.8) and (2.10) to eliminate λ_4, we obtain

$$[(1 + \gamma) N_{13}^2 - \gamma] =$$

$$= \frac{F}{F_1 + F_2}[(1 + \gamma) N_1^2 - \gamma]\left(\frac{1 - \lambda_5''^2\gamma}{1 - \lambda_2^2\gamma}\right)\frac{\lambda_2}{\lambda_5''}. \quad (2.11)$$

The values of N_{13} and λ_5'' are determined from the solution of the system of Eqs. (2.9) and (2.11). The remaining parameters may be found from the conditions at the primary shock wave.

(d) In this case determining the stream parameters in the volume reduces to one of the problems considered in paper [2]. Using the conditions of dynamic compatibility at the primary shock wave, the conditions at the rarefaction wave in the pipe, as well as the continuity equation, we obtain the following system of equations for M_3, M_5'', and N_{13}, the solution of which is determined by the stream parameters in the volume:

$$\frac{F}{F_1 + F_2} = \frac{M_5''}{M_3}\left(\frac{1 + \alpha M_3^2}{1 + \alpha M_5''^2}\right)^{1/2\varkappa},$$

$$[(1 + \gamma) N_{13}^2 - \gamma] = \left(\frac{1 + \alpha M_3^2}{1 + \alpha M_5''^2}\right)^{k/(k-1)}\left(\frac{1 + \alpha M_2}{1 + \alpha M_3}\right)^{k/\alpha}$$

$$\frac{2}{k+1}\left(N_{13} - \frac{1}{N_{13}}\right) =$$

$$= M_5''\left(\frac{1 + \alpha M_2}{1 + \alpha M_3}\right)\left(\frac{1 + \alpha M_3^2}{1 + \alpha M_5''^2}\right)^{1/2} a_2, \quad \alpha = \frac{k-1}{2}. \quad (2.12)$$

3. After the primary shock wave reaches the bottom of the volume, it is reflected and begins to propagate up stream interacting with the contact surface and the secondary shock wave (if one is formed). The results of this interaction may be calculated from the relations given in paper [3]. In what follows we consider the propagation of the reflected shock wave in the gap from the moment when it arrives at the end plane of the pipe. If the problem is strictly posed, then in order to find the flow parameters in the gap behind the shock wave we must integrate a system of three partial differential equations with extremely complicated boundary conditions. At present such a problem may only be solved numerically. In the present paper the problem is solved on the following assumptions: a) the presence of a contact surface between the reflected shock wave which has passed into the pipe and the end plane of the pipe is neglected; b) the lengthwise change of gas parameters in the region between the end plane of the pipe and the shock wave occurs instantaneously; c) there are no wave-like processes in the volume between the end of the pipe and the bottom of the volume, and the volume is filled in a quasi-stationary manner. Figure 2 gives the flow pattern and the notation used in solving the problem.

The gas from the volume flows into the gap with a flow rate $Q_2 = \rho_7 u_7 F_2$, where u_7, ρ_7 are the gas velocity and density, respectively, behind the shock wave and Q_1 is the rate of flow of gas from the pipe. The equation of mass conservation for the volume V is then written in the following manner:

$$V d\rho_6 / dt = Q_1 - Q_2 \, , \tag{3.1}$$

where ρ_6 is the mean density at the time t in the volume V.

We now express ρ_6 and u_7 in (3.1) in terms of ρ_7. The equations of conservation of mass, momentum, and energy for the control surfaces situated between the shock wave and the end plane of the pipe as well as in the volume V have the form

$$u_6\rho_6 F_1 = u_7\rho_7 F_2 = \theta,$$
$$\rho_6 u_6^2 F_1 + p_6 F_1 = \rho_7 u_7^2 F_2 + p_7 F_2 = \Phi,$$
$$\rho_6 u_6 F_1\left(\frac{1}{k-1}\frac{p_6}{\rho_6} + \frac{u_6^2}{2}\right) = \rho_7 u_7 F_2\left(\frac{1}{k-1}\frac{p_7}{\rho_7} + \frac{u_7^2}{2}\right) = \psi \, . \tag{3.2}$$

Solving this system we have

$$\rho_6 = \frac{\theta\Phi + [\theta^2\Phi^2 - 4(k-1)(2-k)\theta^3\psi]^{1/2}}{2F_1\psi(k-1)} \, . \tag{3.3}$$

Here p_7 and u_7 are expressed in terms of ρ_7 using the conditions of dynamic compatibility

$$p_7 = p_4\frac{\varkappa\rho_7/\rho_4' - 1}{\varkappa - \rho_7/\rho_4'}\, , \qquad u_7 = \left[a_4'\left(\frac{\rho_7}{\rho_4'} - 1\right) R \mp u_4'\right]\frac{\rho_4'}{\rho_7}\, ,$$
$$R = \left(\frac{1}{\varkappa - \rho_7/\rho_4'}\frac{2}{k-1}\frac{\rho_7}{\rho_4'}\right)^{1/2},$$

(minus for $p_4 < 1$, plus for $p_4 > 1$). Differentiating (3.3) with respect to ρ_7 we obtain

$$d\rho_6 = \{AC + (2AB - Eu_7^3\rho_7^2[4u_7'\rho_7 p_7 + 3u_7 p_7 + u_7\rho_7 p_7' + (k-1)(3u_7^2\rho_7^2 u_7' + 2u_7^3\rho_7)])C/(2H) - (B+H)[(u_7' p_7 + u_7 p_7')/(k-1) + (3u_7^2 u_7'\rho_7 + u_7^3)/2]\} F_2/(2CF_1)\, d\rho_7 \, ,$$

$$A = \frac{3u_7^2 u_7'\rho_7^2 + 2u_7^3\rho_7 + u_7'\rho_7 p_7 + u_7 p_7 + u_7\rho_7 p_7'}{k-1}\, ,$$

$$B = \frac{u_7^3\rho_7^2 + p_7 u_7\rho_7}{k-1}\, ,$$

$$C = \frac{u_7 p_7}{k-1} + \frac{u_7^3\rho_7}{2}\, , \qquad D = \frac{u_7^4\rho_7^3 p_7}{k-1} + \frac{u_7^6\rho_7^4}{2}\, ,$$

$$E = \frac{4(2-k)}{(k-1)^2}\, , \qquad H = \left(B^2 - 4\frac{2-k}{k-1}D\right)^{1/2},$$

$$p_7' = [\varkappa(\varkappa - \rho_7/\rho_4') + (\varkappa\rho_7/\rho_4' - 1)](\varkappa - \rho_7/\rho_4')^{-2} p_4\, ,$$

$$u_7' = \{[R + (\rho_7/\rho_4' - 1)\varkappa R^{-1}(k-1)^{-1} \times (\varkappa - \rho_7/\rho_4')^{-2}]\rho_7/\rho_4' - [(\rho_7/\rho_4' - 1)R - u_4']\}(\rho_4'/\rho_7)^2 \, . \tag{3.4}$$

Using (3.4) to replace the value of $d\rho_6$ in (3.1) and integrating, we obtain

$$t = \int\limits_{\rho_{7н}}^{\rho_7} \frac{Vf(\rho_7)\, d\rho_7}{(Q_1 - Q_2)}\, , \tag{3.5}$$

where $\rho_{7н}$ is the gas density behind the shock wave which has passed into the gap at the moment when it passed the end plane of the pipe; $f(\rho_7)$ is the factor in front of $d\rho_7$ in formula (3.4). The moment when the reflected shock wave passes the end plane of the pipe is chosen as the time origin. The quantity $\rho_{7н}$ is determined from the flow rate equation at the initial moment when $t = 0$; $u_6\rho_6 F_1 = u_{7н}\rho_{7н}F_2$. At the front of the shock wave which has passed into the gap we have

$$\rho_4'(N_{21} \pm u_4') = \rho_{7н}(N_{21} - u_{7н})\, , \tag{3.6}$$

where N_{21} is the velocity of shock wave propagation in the gap; the minus sign is taken for $p_4 < 1$, the plus sign for $p_4 > 1$. Using the relation for $u_7(\rho_7)$ at the shock wave and also the expression (3.6), from the condition that the flow rates should be equal we have

$$\frac{u_6\rho_6 F_1}{\rho_{7н}F_2} = a_4'\left(\frac{\rho_{7н}}{\rho_4'} - 1\right)\frac{R_н\rho_4'}{\rho_7} - \frac{u_4'\rho_4'}{\rho_{7н}}\, ,$$
$$R_н = \left(\frac{1}{\varkappa - \rho_{7н}/\rho_4'}\frac{2}{k-1}\frac{\rho_{7н}}{\rho_4'}\right)^{1/2}. \tag{3.7}$$

Here u_6 and ρ_6 are the values at the shock wave which has arrived at the end plane of the pipe after being reflected from the bottom of the volume.

The integral (3.5) is evaluated numerically. The integration leads to the function $\rho_7(t)$. The remaining gas parameters behind the shock wave, as well as the law of motion of the shock wave in the gap, are determined from the conditions of dynamic compatibility at the shock wave front.

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